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Martingale models for quantum state reduction

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Abstract

Stochastic models for quantum state reduction give rise to statistical laws that are in most respects in agreement with those of quantum measurement theory. Here we examine the correspondence of the two theories in detail, making a systematic use of the methods of martingale theory. An analysis is carried out to determine the magnitude of the fluctuations experienced by the expectation of the observable during the course of the reduction process and an upper bound is established for the ensemble average of the greatest fluctuations incurred. We consider the general projection postulate of Lüders applicable in the case of a possibly degenerate eigenvalue spectrum, and derive this result rigorously from the underlying stochastic dynamics for state reduction in the case of both a pure and a mixed initial state. We also analyse the associated Lindblad equation for the evolution of the density matrix, and obtain an exact time-dependent solution for the state reduction that explicitly exhibits the transition from a general initial density matrix to the Lüders density matrix. Finally, we apply Girsanov's theorem to derive a set of simple formulae for the dynamics of the state in terms of a family of geometric Brownian motions, thereby constructing an explicit unravelling of the Lindblad equation.

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1. Introduction

According to nonrelativistic quantum mechanics, the evolution of the state of an isolated quantum system is described by a deterministic unitary transformation, governed by the Schrödinger equation. The behaviour of the state of a quantum system following the measurement of an observable is less well understood, however, and has been the subject of much debate. In quantum measurement theory it is usually assumed that when the measurement of an observable with a discrete spectrum is carried out on a system prepared in a prescribed

initial state, then the state reduces randomly to one of the eigenstates of the observable being measured. This is the so-called projection postulate of von Neumann [1], which was later generalized by Lüders [2] to handle the case of a measurement of an observable with a degenerate spectrum. The Lüders postulate has the virtue of being unambiguously applicable whether or not the initial state is pure, and whether or not the eigenvalue spectrum is nondegenerate.

Stochastic extensions of the Schrödinger equation have been increasingly attracting attention as plausible dynamical models for state vector reduction in quantum mechanics [3–9]. In such models, the Schrödinger equation is generalized to take the form of a special type of stochastic differential equation on Hilbert space. For example, given a Hamiltonian \hat{H} and a commuting observable \hat{F} with a discrete spectrum, there exists a natural stochastic differential equation generalizing the Schrödinger equation with the property that starting from a given initial pure state, the system evolves randomly in such a way that asymptotically it reaches one of the eigenstates of \hat{F} with the correct quantum probability. More generally, given a compatible family of observables \hat{F}_α ($\alpha = 1, 2, \dots, r$), each of which commutes with \hat{H} , a similar result holds, with an asymptotic reduction to one of the common eigenstates of the given family of observables. For recent reviews outlining the development of this approach, with extensive references, see [10] and [11].

The purpose of this paper is to analyse in some detail the statistical laws associated with stochastic extensions of the Schrödinger equation, and to show in particular how the projection postulate in the general form due to Lüders can be derived as a consequence of the dynamics. The plan of the paper is as follows. In sections 2 and 3 we review the von Neumann and Lüders versions of the projection postulate. In section 4 we present a reasonably self-contained account of the basic principles of stochastic state reduction, along with a brief synopsis of the relevant mathematical tools of stochastic analysis. For the purposes of illustration we consider primarily the case of an energy-based reduction model, for which the associated dynamics are given by the stochastic differential equation (11), though most of the relevant mathematical and physical ideas can be readily generalized to the class of reduction processes noted above, based on a compatible family of observables that commute with the Hamiltonian.

Throughout the discussion we emphasize the role of martingale methods as an aid to the advancement of our understanding of quantum phenomena. In particular we show that the expectation of the Hamiltonian, which fluctuates during the course of the reduction process, is a martingale, and its variance is a supermartingale. The martingale property satisfied by the expectation of the Hamiltonian can be viewed as a kind of weak conservation law for the energy, generalizing the Ehrenfest relation. In section 5 we use the Doob–Kolmogorov maximal inequalities to obtain a set of upper and lower bounds on the fluctuations of the energy during the reduction process, and show that the magnitude of a typical fluctuation is roughly of the order of the initial energy uncertainty. We also demonstrate that the process followed by the squared energy uncertainty is given by the conditional variance of the terminal value of the energy.

In sections 6 and 7 we derive the projection postulate, and prove that the collapse to the general Lüders state occurs with the appropriate probability, whether or not the initial state is pure and whether or not the eigenvalue spectrum is degenerate. Then in section 8 we study the implied evolution of the density matrix for a given initial state, not necessarily pure, and derive an exact time-dependent solution for the Lindblad equation associated with the reduction. We conclude in section 9 by introducing a change-of-measure technique to solve the stochastic differential equation for the dynamics of the state vector, thereby constructing an explicit unravelling of the Lindblad equation.

2. Von Neumann projection

Let us consider a quantum system which, for mathematical simplicity, we shall assume is characterized by a finite dimensional Hilbert space \mathcal{H} of dimension N . Suppose that F is an observable for which the corresponding Hermitian operator acting on \mathcal{H} is denoted \hat{F} . Our interest here is in providing a clearer understanding of what happens if the observable F is measured when the system is in a given pure state, corresponding to a ray through the origin in \mathcal{H} .

First we shall examine the more straightforward case where F has a nondegenerate spectrum, and the eigenvalues of \hat{F} are given by the numbers f_n for $n = 1, 2, \dots, N$, with the property that $f_n \neq f_m$ for $n \neq m$. The eigenstate corresponding to the eigenvalue f_n will be denoted $|f_n\rangle$, so that $\hat{F}|f_n\rangle = f_n|f_n\rangle$.

Let us write $|\psi_0\rangle$ for a representative state vector for the given initial pure state. We shall assume that $|\psi_0\rangle$ is normalized to unity, so $\langle\psi_0|\psi_0\rangle = 1$. We shall also assume that $\langle f_n|f_m\rangle = \delta_{nm}$.

According to the *projection postulate*, when a measurement of F is made, the state vector undergoes a transition $|\psi_0\rangle \mapsto |f_n\rangle$ to one of the eigenstates of \hat{F} . This occurs for a specified value of n with the probability

$$\pi_n = |\langle f_n|\psi_0\rangle|^2 \quad (1)$$

and the result of the measurement in that case is the eigenvalue f_n . The associated transition $|\psi_0\rangle \mapsto |f_n\rangle$ is called the state reduction or ‘collapse of the wavefunction’ arising from the measurement of F .

A more precise way of stating this is that when the observable F is measured, the initial pure state $|\psi_0\rangle$ transforms to a ‘mixture’, i.e. a random state $|f\rangle$ with the property that $|f\rangle$ is given by the eigenstate $|f_n\rangle$ with the probability π_n . The fact that $|f\rangle$ is a mixture reflects our ignorance of what the result of the measurement process will be. When the observable F is measured, we can be confident that the result is one of the values f_n , and that the new state is the eigenstate represented by $|f_n\rangle$, but we cannot say in advance which one it will be. This is what is meant by saying that the result of the measurement is random.

The density matrix $\hat{\rho}$ associated with the random state $|f\rangle$ and the probability distribution π_n is given by the expectation of the random projection operator $|f\rangle\langle f|$ associated with $|f\rangle$. In other words, we have

$$\begin{aligned} \hat{\rho} &= \mathbb{E}[|f\rangle\langle f|] \\ &= \sum_{n=0}^N \pi_n |f_n\rangle\langle f_n|. \end{aligned} \quad (2)$$

Here \mathbb{E} denotes expectation with respect to the distribution π_n . The mixture $|f\rangle$ typically carries more information than the density matrix $\hat{\rho}$ alone, because if one is given $\hat{\rho}$, then in general there are many different mixtures that correspond to it [12].

The significance of the density matrix $\hat{\rho}$ is that if G is any other observable, not necessarily compatible with F , and we measure G after we measure F , then the expected value of G is

$$\begin{aligned} \mathbb{E}[\langle f|\hat{G}|f\rangle] &= \sum_n \pi_n \langle f_n|\hat{G}|f_n\rangle \\ &= \text{Tr} \sum_n \pi_n |f_n\rangle\langle f_n|\hat{G} \\ &= \text{Tr} \hat{\rho} \hat{G} \end{aligned} \quad (3)$$

where Tr denotes the trace operation.

An alternative way of facilitating the description of the density matrix associated with the measurement outcome for an observable F is to introduce the projection operator $\hat{P}_n = |f_n\rangle\langle f_n|$ associated with the eigenvalue f_n . When F is measured, the state $|\psi_0\rangle$ is transformed to $\pi_n^{-1/2}\hat{P}_n|\psi_0\rangle$ if the measurement outcome is known to be f_n , for which the corresponding probability is π_n . An analogous transformation holds for the corresponding density matrix $\hat{\rho}_0 = |\psi_0\rangle\langle\psi_0|$ which transforms according to the scheme

$$\begin{aligned}\hat{\rho}_0 \mapsto \hat{\rho}_\infty &= \pi_n^{-1}\hat{P}_n\hat{\rho}_0\hat{P}_n \\ &= |f_n\rangle\langle f_n|.\end{aligned}\quad (4)$$

Here we use the notation $\hat{\rho}_0$ to signify the density matrix before the measurement, and $\hat{\rho}_\infty$ to signify the density matrix after the measurement.

More generally, if F is measured but the outcome is *not* known, then $\hat{\rho}_0$ transforms according to the scheme

$$\begin{aligned}\hat{\rho}_0 \mapsto \hat{\rho}_\infty &= \sum_{n=1}^N \hat{P}_n\hat{\rho}_0\hat{P}_n \\ &= \sum_{n=1}^N |f_n\rangle\langle f_n|\psi_0\rangle\langle\psi_0|f_n\rangle\langle f_n| \\ &= \sum_{n=1}^N \pi_n |f_n\rangle\langle f_n|.\end{aligned}\quad (5)$$

The density matrix itself is often referred to as representing the ‘state’ of a quantum system. This is because the density matrix contains all the information required to calculate ensemble expectations and probabilities for the measurement outcomes of quantum observables, conditional on the present state of knowledge of the observer. Therefore, different mixtures yielding the same density matrix are equivalent as far as observations are concerned.

In the examples above, the ensemble interpretation is as follows. We prepare a large number of independent identical quantum systems each in the state $|\psi_0\rangle$ and then measure F . Now there are two possibilities. In the first case, we only keep those systems for which the result of the measurement of F was the value f_n . The new ensemble then consists of a large number of independent systems each of which is in the pure state $|f_n\rangle$. The corresponding density matrix is $|f_n\rangle\langle f_n|$. In the second case, we keep all the systems after F has been measured. The resulting ensemble is therefore a mixture, and for each value of n , a given system is in the pure state $|f_n\rangle$ with probability π_n . The corresponding density matrix is then given by (5).

3. Lüders’ postulate

Let us now turn to the case of an observable with a degenerate spectrum. In this case, we shall write $|f_{n,j}\rangle$ for an orthogonal basis of distinct eigenstates of \hat{F} sharing the same eigenvalue f_n . Here $n = 1, 2, \dots, D$, where D is the number of distinct energy levels, and $j = 1, 2, \dots, d_n$, where d_n is the dimensionality of the subspace \mathcal{H}_n of \mathcal{H} spanned by the eigenstates with eigenvalue f_n . For convenience we normalize $|f_{n,j}\rangle$ such that

$$\langle f_{n,j}|f_{m,k}\rangle = \delta_{nm}\delta_{jk}.\quad (6)$$

Then the projection operator \hat{P}_n onto the subspace spanned by states for which $F = f_n$ is given by

$$\hat{P}_n = \sum_{j=1}^{d_n} |f_{n,j}\rangle\langle f_{n,j}|. \quad (7)$$

We note that \hat{P}_n is independent of the specific choice of basis made for the designated subspace, and that $\hat{P}_n \hat{P}_m = \hat{P}_n \delta_{nm}$ and $\hat{P}_n |f_{m,k}\rangle = \delta_{nm} |f_{m,k}\rangle$.

With these preliminaries in mind, suppose the observable F is measured when the system is in the pure state $|\psi_0\rangle$, and the result of the measurement is one of the degenerate eigenvalues f_n . In this case, it is perhaps less obvious *a priori* what the correct probability is for the outcome, or indeed what becomes of the state once the measurement result is known. A more refined version of the projection postulate is required to deal with this situation, due to Lüders [2], according to which the measurement outcome probability is

$$\text{Prob}[F = f_n] = \langle \psi_0 | \hat{P}_n | \psi_0 \rangle \quad (8)$$

and the associated state reduction is given by

$$|\psi_0\rangle \mapsto \pi_n^{-1/2} \hat{P}_n |\psi_0\rangle \quad (9)$$

where

$$\begin{aligned} \pi_n &= \langle \psi_0 | \hat{P}_n | \psi_0 \rangle \\ &= \sum_{j=1}^{d_n} |\langle f_{n,j} | \psi_0 \rangle|^2. \end{aligned} \quad (10)$$

Thus, of all the possible eigenstates with eigenvalue f_n , a single choice is made, given by the projection from the initial state vector onto the relevant subspace. As in the nondegenerate case, the measurement outcome can be described by a mixture, where the random state $|f\rangle$ is given by the normalized Lüders state $|f_n\rangle = \pi_n^{-1/2} \hat{P}_n |\psi_0\rangle$ with probability π_n ($n = 1, 2, \dots, D$).

The validity of the Lüders postulate can, in principle, be tested by a succession of measurements of the energy of a system, followed by the measurement of another observable incompatible with the energy. Consider for example the system consisting of a pair of noninteracting spin- $\frac{1}{2}$ particles in an external magnetic field aligned along the z -axis, for which the energy eigenstates are the spin-0 singlet and spin-1 triplet. The corresponding eigenvalues are degenerate and are given by $E_1 = -1$, $E_2 = E_3 = 0$, and $E_4 = 1$, in suitable units. We choose the initial state such that the Lüders state associated with the degenerate energy eigenvalue is given by the spin-0 singlet. Then, for an ensemble of identically prepared systems, we measure the energy, and discard those systems for which the outcome is given by one of the eigenvalues ± 1 . The remaining systems, according to Lüders, are in the spin-0 singlet state, whereas according to von Neumann [1], these states would in general be in superpositions of the singlet and $S_z = 0$ triplet states, the precise details of which depend on the nature of the measurement apparatus. In the present set-up, one can measure the total spin operator to determine the outcome of the initial energy measurement. This is because the result of the total spin gives the eigenvalue 0 if and only if the system is in the singlet state.

An important feature of the Lüders postulate is the inherent ‘instability’ of the reduction process implied for certain types of measurements. That is, in the case of an observable with a degenerate eigenvalue f_n , the projection is onto a single state $\pi_n^{-1/2} \hat{P}_n |\psi_0\rangle$; whereas if the observable is perturbed even slightly, breaking the degeneracy and producing, say, two distinct but close eigenvalues f_n and $f_{n'}$, then the reduction process bifurcates, leading to one or the other of two orthogonal, and hence maximally separated, eigenstates $|f_n\rangle$ and $|f_{n'}\rangle$.

Thus we are led to consider that if the phenomenon of state reduction itself arises as a consequence of a *dynamical process*, then this process must have sufficiently special properties to ensure that under a smooth deformation of the parameters characterizing the observable being measured, the resulting dynamics exhibits the required discontinuous behaviour and produces the corresponding bifurcation in the postulated transition probabilities. In what follows, we shall demonstrate that the standard stochastic models for quantum state reduction exhibit the required special properties, including the relevant bifurcation phenomena. In short, we can *derive* the projection postulate from a dynamical model, including the specifics of the Lüders postulate and its consequences in the case of a degenerate spectrum and a mixed initial state.

4. State vector reduction

In this section we consider in some detail the case of the standard energy-based stochastic extension of the Schrödinger equation, for which, if we set $\hbar = 1$, the dynamics are given by the following stochastic differential equation on \mathcal{H} :

$$d|\psi_t\rangle = -i\hat{H}|\psi_t\rangle dt - \frac{1}{8}\sigma^2(\hat{H} - H_t)^2|\psi_t\rangle dt + \frac{1}{2}\sigma(\hat{H} - H_t)|\psi_t\rangle dW_t. \quad (11)$$

The properties of dynamical processes of this type have been investigated by a number of authors [4, 6–9, 13]. In particular, as we shall demonstrate with various examples, many of the familiar probabilistic features of standard quantum mechanics, including the Born rules, can be *deduced* from (11), or suitable generalizations thereof. Here, $|\psi_t\rangle$ denotes the random state vector at time t , for which an initial state $|\psi_0\rangle$ is prescribed. For the moment we consider the case when $|\psi_0\rangle$ is known, and later we turn to the case where $|\psi_0\rangle$ is random. For the expectation of the Hamiltonian operator \hat{H} in the state $|\psi_t\rangle$ we write

$$H_t = \frac{\langle\psi_t|\hat{H}|\psi_t\rangle}{\langle\psi_t|\psi_t\rangle} \quad (12)$$

from which it follows that H_t itself can be interpreted as a random process, which we shall call the *energy process*. Strictly speaking, an expression such as $(\hat{H} - H_t)^2|\psi_t\rangle$ in (11) should be written $(\hat{H} - H_t\hat{1})^2|\psi_t\rangle$, where $\hat{1}$ is the identity operator, but there will be no ambiguity if we use the more compact notation.

The energy-based stochastic extension of the Schrödinger equation (11) is of great interest because it represents perhaps the simplest plausible model for the collapse of the wave function, and as such exhibits many remarkable features, both physically and mathematically. It will be useful if we begin our analysis with a brief overview of the probabilistic framework implicit in the characterization of (11).

The stochastic differential equation for the process $|\psi_t\rangle$ is to be understood as defined on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration \mathcal{F}_t , with respect to which W_t is a standard Wiener process (Brownian motion). Here Ω is the sample space, \mathcal{F} is a σ -field on Ω , and \mathbb{P} is a probability measure on \mathcal{F} .

The filtration \mathcal{F}_t represents the information available at time t , where $t \in [0, \infty)$. More precisely, a filtration of \mathcal{F} is a collection \mathcal{F}_t ($0 \leq t < \infty$) of σ -subfields of \mathcal{F} with the property that $s \leq t$ implies $\mathcal{F}_s \subset \mathcal{F}_t$. A function $X : \Omega \mapsto \mathbb{R}$ is said to be *measurable* with respect to \mathcal{F} if for each $x \in \mathbb{R}$ the set consisting of all $\omega \in \Omega$ satisfying $X(\omega) \leq x$ is an element of \mathcal{F} . This assures that $\text{Prob}[X \leq x]$ exists with respect to the given measure \mathbb{P} on \mathcal{F} , and we say that X is a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then by a random process we mean a parametrized family of random variables X_t ($0 \leq t < \infty$) on $(\Omega, \mathcal{F}, \mathbb{P})$. With a slight abuse of notation we let X_t stand both for the entire process X_t ($0 \leq t < \infty$), as well as the random variable X_t for some given value of t ; usually it will be evident from context which meaning

is intended. Likewise \mathcal{F}_t may denote the entire filtration \mathcal{F}_t ($0 \leq t < \infty$), or the σ -subfield of \mathcal{F} corresponding to the information set at time t . If a random process X_t is such that for each value of t the corresponding random variable X_t is \mathcal{F}_t -measurable, then we say that the process X_t is *adapted* to the filtration \mathcal{F}_t . The idea of ‘adaptedness’ is important because it is through this device that a notion of causality is introduced for the class of process we consider.

It should be emphasized that the probabilistic concepts outlined here and in what follows are introduced not merely for the sake of mathematical clarity (although this is in itself a desirable feature), but also because it makes possible the use of various powerful analytical tools, examples of which we shall discuss shortly.

The concept of conditional expectation plays a particularly important role in the theory of quantum state reduction, and hence it will be helpful if we elaborate slightly on the idea here. For any random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$ we define its expectation

$$\mathbb{E}[X] = \int_{\Omega} X(\omega) d\mathbb{P}(\omega) \quad (13)$$

by use of the Lebesgue integral. One can think of $\mathbb{E}[X]$ as the ensemble average of X . Then if \mathcal{E} is a σ -subfield of \mathcal{F} , the random variable Y is said to be (a version of) the conditional expectation of X with respect to \mathcal{E} if Y is \mathcal{E} -measurable and if $\mathbb{E}[X1_A] = \mathbb{E}[Y1_A]$ for all sets $A \in \mathcal{E}$. In that case we write $Y = \mathbb{E}[X|\mathcal{E}]$. Here 1_A denotes the indicator function for the set A , so $1_A(\omega) = 1$ for $\omega \in A$ and $1_A(\omega) = 0$ for $\omega \notin A$. By convention we write $\mathbb{E}_t[X] = \mathbb{E}[X|\mathcal{F}_t]$ for the conditional expectation of X given information up to time t . One can think of $\mathbb{E}_t[X]$ as the ensemble average of X conditional on the history of events up to time t being specified.

A useful result that follows on from these definitions is the so-called ‘tower property’ of conditional expectation, which says that if \mathcal{D} is a σ -subfield of \mathcal{E} then $\mathbb{E}[\mathbb{E}[X|\mathcal{E}]|\mathcal{D}] = \mathbb{E}[X|\mathcal{D}]$. If we set $\mathcal{D} = (\emptyset, \Omega)$, the smallest σ -subfield of \mathcal{F} , then $\mathbb{E}[X|\mathcal{D}] = \mathbb{E}[X]$. This is because the only \mathcal{D} -measurable random variables in that case are the constants: if $X(\omega)$ is constant on Ω , then for any given x we have $X(\omega) \leq x$ either for all $\omega \in \Omega$ or for no $\omega \in \Omega$; conversely, if $X(\omega)$ is not constant, then we can find a value of x and two points $\omega_1, \omega_2 \in \Omega$ such that $X(\omega_1) \leq x$ and $X(\omega_2) > x$, which shows that $X(\omega)$ is not \mathcal{D} -measurable—that is to say, the set $\{\omega : X(\omega) \leq x\}$ is not an element of \mathcal{D} . It follows then from the tower property that $\mathbb{E}[\mathbb{E}[X|\mathcal{E}]] = \mathbb{E}[X]$, the so-called *law of total probability*. In the case of a filtration \mathcal{F}_t ($0 \leq t < \infty$), if we take $\mathcal{E} = \mathcal{F}_t$ and $\mathcal{D} = \mathcal{F}_s$, then the tower property reads $\mathbb{E}_s[\mathbb{E}_t[X]] = \mathbb{E}_s[X]$ for $s \leq t$; whereas the law of total probability implies that $\mathbb{E}[\mathbb{E}_t[X]] = \mathbb{E}[X]$ for all $t \geq 0$.

Now suppose X_t ($0 \leq t < \infty$) is an adapted process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration \mathcal{F}_t ($0 \leq t < \infty$). Then we say X_t is a *martingale* if the following two conditions hold: $\mathbb{E}[|X_t|] < \infty$ for all $0 \leq t < \infty$, and $\mathbb{E}_s[X_t] = X_s$ for all $0 \leq s \leq t < \infty$. If instead of the latter condition X_t satisfies $\mathbb{E}_s[X_t] \geq X_s$ then we say X_t is a *submartingale*, and if $\mathbb{E}_s[X_t] \leq X_s$ we say X_t is a *supermartingale*.

Returning now to our investigation of the process (11), we note that the parameter σ governs the rate at which the state vector reduction proceeds for a given level of initial uncertainty in the energy. The units of σ are

$$\sigma \sim [\text{energy}]^{-1}[\text{time}]^{-1/2}. \quad (14)$$

The characteristic timescale τ_R associated with the collapse of the wavefunction is

$$\tau_R = \frac{1}{\sigma^2 V_0} \quad (15)$$

where V_0 is the square of the initial energy uncertainty ΔH .

In what follows we shall make no specific assumptions about the value of σ . Nevertheless, we note, as is discussed in [7, 8], that if $\sigma \sim M_p^{-1/2}$ in microscopic units with $\hbar = c = 1$,

where M_p is the Planck mass, then the ‘large numbers’ cancel out and we are left with a typical reduction timescale of

$$\tau_R \sim \left(\frac{2.8 \text{ MeV}}{\Delta H} \right)^2 \text{ s.} \quad (16)$$

This expression is interesting as a candidate for τ_R inasmuch as it relates energy spreads typical of atomic and nuclear phenomena to timescales that are accessible in the laboratory.

The factor of $\frac{1}{2}$ appearing in front of σ in (11) is for convenience, and ensures consistency with the notation of [7–9].

It follows from the Ito rules $(dt)^2 = 0$, $dt dW_t = 0$, and $(dW_t)^2 = dt$, as well as the special form of the nonlinear terms appearing in (11), that the norm of the state $|\psi_t\rangle$ is preserved under the evolution (11). This can be seen as follows. The Ito product rule states that if X_t and Y_t are Ito processes then $d(X_t Y_t) = Y_t dX_t + X_t dY_t + dX_t dY_t$. As a consequence, we have

$$d(\langle \psi_t | \psi_t \rangle) = (d\langle \psi_t |) |\psi_t\rangle + \langle \psi_t | (d|\psi_t\rangle) + (d\langle \psi_t |) (d|\psi_t\rangle). \quad (17)$$

Now the Hermitian conjugate of (11) is

$$d\langle \psi_t | = i\langle \psi_t | \hat{H} dt - \frac{1}{8}\sigma^2 \langle \psi_t | (\hat{H} - H_t)^2 dt + \frac{1}{2}\sigma \langle \psi_t | (\hat{H} - H_t) dW_t. \quad (18)$$

Therefore, by use of the Ito rules, we obtain

$$(d\langle \psi_t |) |\psi_t\rangle = (iH_t - \frac{1}{8}\sigma^2 V_t) \langle \psi_t | \psi_t \rangle dt \quad (19)$$

and its conjugate,

$$\langle \psi_t | (d|\psi_t\rangle) = (-iH_t - \frac{1}{8}\sigma^2 V_t) \langle \psi_t | \psi_t \rangle dt \quad (20)$$

together with

$$(d\langle \psi_t |) (d|\psi_t\rangle) = \frac{1}{4}\sigma^2 V_t \langle \psi_t | \psi_t \rangle dt \quad (21)$$

where V_t is given by formula (23) below. It follows then from (17) that $d(\langle \psi_t | \psi_t \rangle) = 0$, as desired. This result is useful in calculations because we can assume the initial norm to be unity, without loss of generality, and thus $\langle \psi_t | \psi_t \rangle = 1$ for all t .

An analogous calculation shows that the energy process H_t defined in (12) satisfies

$$dH_t = \sigma V_t dW_t \quad (22)$$

where V_t is the process for the variance (squared uncertainty) of \hat{H} in the state $|\psi_t\rangle$, given by

$$V_t = \frac{\langle \psi_t | (\hat{H} - H_t)^2 | \psi_t \rangle}{\langle \psi_t | \psi_t \rangle}. \quad (23)$$

The variance process for the Hamiltonian has the property that $V_t = 0$ at time t if and only if $|\psi_t\rangle$ is an energy eigenstate at that time. As a consequence of (22), we can write

$$H_t = H_0 + \sigma \int_0^t V_u dW_u \quad (24)$$

where H_0 is the initial expectation value for the energy. Now it is a general property of the stochastic integral that for any \mathcal{F}_t -adapted integrand A_t satisfying $\mathbb{E}[\int_0^t A_u^2 du] < \infty$ we have

$$\mathbb{E}_s \left[\int_0^t A_u dW_u \right] = \int_0^s A_u dW_u \quad (s \leq t). \quad (25)$$

The variance process V_t is bounded by $\frac{1}{4}(E_+ - E_-)^2$ where E_+ and E_- are the largest and the smallest energy eigenvalues, respectively, which implies that $\int_0^t V_u^2 du < \infty$. Furthermore, we note that $|H_t|$ is bounded by $\max(|E_+|, |E_-|)$. It follows that H_t is a martingale:

$$\mathbb{E}_s[H_t] = H_s \quad (s \leq t). \quad (26)$$

The martingale condition is the stochastic analogue of a conservation law, and thus (26) can be interpreted as a *weak conservation law for the energy*. We recall that, for ordinary quantum mechanical evolution in the case of a time-independent Hamiltonian, the Schrödinger equation $\partial_t |\psi_t\rangle = -i\hat{H}|\psi_t\rangle$ ensures that the expectation of the Hamiltonian $H_t = \langle \psi_t | \hat{H} | \psi_t \rangle / \langle \psi_t | \psi_t \rangle$ is conserved along the Schrödinger trajectories. In the case of the stochastic extension of the Schrödinger equation we have instead the martingale relation (26) which ensures that the ensemble average of the energy is conserved.

Because (26) plays a pivotal role in understanding the nature of the reduction process, we shall sometimes refer to the system of stochastic dynamics described by (11) as a *martingale model* for quantum state reduction.

We note, more generally, that if the operator $\hat{G} = g(\hat{H})$ is given by a function of \hat{H} , then the process

$$G_t = \frac{\langle \psi_t | \hat{G} | \psi_t \rangle}{\langle \psi_t | \psi_t \rangle} \tag{27}$$

is also weakly conserved, i.e. $\mathbb{E}_s[G_t] = G_s$ for $s \leq t$. Thus, for example, if $g(x) = x^n$ and we introduce the notation

$$H_t^{(n)} = \frac{\langle \psi_t | \hat{H}^n | \psi_t \rangle}{\langle \psi_t | \psi_t \rangle} \tag{28}$$

for the n th moment of the energy, then

$$dH_t^{(n)} = \sigma(H_t^{(n+1)} - H_t H_t^{(n)}) dW_t \tag{29}$$

where $H_t = H_t^{(1)}$.

With these formulae in mind, let us consider now the dynamics of the variance process V_t . Writing $V_t = H_t^{(2)} - (H_t)^2$ it follows, according to Ito's lemma, that

$$dV_t = dH_t^{(2)} - 2H_t dH_t - (dH_t)^2. \tag{30}$$

By use of the Ito rules together with (29) we then deduce that

$$dV_t = -\sigma^2 V_t^2 dt + \sigma \beta_t dW_t \tag{31}$$

where

$$\beta_t = \frac{\langle \psi_t | (\hat{H} - H_t)^3 | \psi_t \rangle}{\langle \psi_t | \psi_t \rangle} \tag{32}$$

is the *skewness* of the energy distribution at time t , i.e. the third central moment of the Hamiltonian. More specifically, we have $\beta_t = H_t^{(3)} - 3H_t H_t^{(2)} + 2(H_t)^3$.

Integrating equation (31) for the dynamics of the variance we obtain

$$V_t = V_0 - \sigma^2 \int_0^t V_u^2 du + \sigma \int_0^t \beta_u dW_u \tag{33}$$

from which it follows at once, by use of (25), that

$$\mathbb{E}_s[V_t] = V_s - \sigma^2 \mathbb{E}_s \left[\int_s^t V_u^2 du \right] \tag{34}$$

and thus

$$\mathbb{E}_s[V_t] \leq V_s \tag{35}$$

for $s \leq t$, which shows that V_t is a *supermartingale*, i.e. a process that on average decreases.

In particular, if we write $\bar{V}_t = \mathbb{E}[V_t]$ for the ensemble average of V_t , then it follows as a special case of (34) that

$$\bar{V}_t = V_0 - \sigma^2 \mathbb{E} \left[\int_0^t V_u^2 du \right]. \quad (36)$$

Differentiating this expression with respect to t we obtain

$$\frac{d\bar{V}_t}{dt} = -\sigma^2 \bar{V}_t^2 (1 + \eta_t) \quad (37)$$

where the process η_t defined by $\eta_t = \mathbb{E}[(V_t - \bar{V}_t)^2]/\bar{V}_t^2$ is non-negative. Therefore, by integration of (37), we obtain

$$\bar{V}_t = \frac{V_0}{1 + \sigma^2 V_0 (t + \xi_t)} \quad (38)$$

where $\xi_t = \int_0^t \eta_s ds$. As a consequence we deduce that

$$\bar{V}_t \leq \frac{V_0}{1 + \sigma^2 V_0 t} \quad (39)$$

which shows that

$$\lim_{t \rightarrow \infty} \bar{V}_t = 0. \quad (40)$$

Alternatively, if we introduce the ‘localization’ process $\Lambda_t = \bar{V}_t^{-1}$ then it follows from (37) that $\partial_t \Lambda_t \geq \sigma^2$, which shows that Λ_t increases without bound [14, 15]. In (39) we see an example of the role of $\tau_R = (\sigma^2 V_0)^{-1}$ as the characteristic timescale of the reduction process. Since V_t is non-negative, it follows that

$$\lim_{t \rightarrow \infty} V_t = 0 \quad (41)$$

almost surely. The dynamical process (11) therefore induces a collapse of the wavefunction, for any choice of the initial state $|\psi_0\rangle$, to an eigenstate of the Hamiltonian.

5. Fluctuation analysis

The martingale property (26) satisfied by the energy process H_t implies in the limit $t \rightarrow \infty$, that $\mathbb{E}[H_\infty] = H_0$. However, the terminal value H_∞ of the energy process, the existence of which we shall establish shortly, is necessarily one of the energy eigenvalues, from which it follows that

$$H_0 = \sum_n \pi_n E_n \quad (42)$$

where π_n is the probability of reaching the eigenstate $|n\rangle$ starting from the given initial state. Therefore, the ensemble average of the measured value of the energy equals the expectation value of the energy in the initial state, as it should.

The importance of this conclusion is that whereas in quantum measurement theory it is essentially an *assumption* that the ‘expectation value’ of an observable in a given state is the ensemble average for the result of a measurement of the observable, in a martingale model one can *prove* that the asymptotic ensemble average agrees with the expectation value, hence justifying the conventional interpretation of this quantity. In particular, using (26) we can write

$$H_t = \mathbb{E}_t[H_\infty] \quad (43)$$

which shows that the quantum expectation value H_t of the observable H at time t is always the best ‘forecast’, based on information currently available, for the outcome of a measurement of H .

A similar result holds for the dispersion of the measured values of the energy. This can be established by use of the Ito isometry. If A_t and B_t are \mathcal{F}_t -adapted real processes that are square-integrable in the sense that $\mathbb{E}[\int_0^t A_s^2 ds] < \infty$ and $\mathbb{E}[\int_0^t B_s^2 ds] < \infty$, then the Ito isometry states that

$$\mathbb{E} \left[\left(\int_0^t A_s dW_s \right) \left(\int_0^t B_s dW_s \right) \right] = \mathbb{E} \left[\int_0^t A_s B_s ds \right]. \quad (44)$$

It follows therefore from (24) that

$$\begin{aligned} \mathbb{E}[(H_t - H_0)^2] &= \sigma^2 \mathbb{E} \left[\left(\int_0^t V_s dW_s \right)^2 \right] \\ &= \sigma^2 \mathbb{E} \left[\int_0^t V_s^2 ds \right] \end{aligned} \quad (45)$$

by virtue of the Ito isometry. By use of expression (33) for V_t we then deduce that

$$\mathbb{E}[(H_t - H_0)^2] = V_0 - \mathbb{E}[V_t]. \quad (46)$$

Taking the limit $t \rightarrow \infty$ and using the fact that $\lim_{t \rightarrow \infty} \bar{V}_t = 0$, we get

$$\begin{aligned} \text{Var}[H_\infty] &= \mathbb{E}[(H_\infty - \mathbb{E}[H_\infty])^2] \\ &= V_0 \end{aligned} \quad (47)$$

which demonstrates that *the variance of the measured energy is in agreement with the squared energy uncertainty in the initial state.*

During the course of the reduction process, the energy H_t of the system can, in principle, deviate far from its initial value H_0 , subject to the condition that it stays in the range $H_t \in [E_-, E_+]$, where E_- and E_+ are the lowest and highest energy levels. Nevertheless, we can show that on average H_t will not deviate too much from H_0 : an upper bound can be set on the maximum fluctuation experienced by the energy, on average, as the reduction proceeds. This bound is given by $2\Delta H$, twice the initial energy uncertainty.

The proof of this result makes use of the Doob–Kolmogorov maximal inequalities (see, e.g., [16], theorem 6.10, or [17], theorem 1.7, p 54). These inequalities state that if M_t is a right-continuous martingale or positive submartingale, and $\mathbb{E}[|M_T|^p] < \infty$ for some $p \geq 1$, then

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |M_t|^p \right] \leq \left(\frac{p}{p-1} \right)^p \mathbb{E} [|M_T|^p] \quad (p > 1) \quad (48)$$

and

$$\text{Prob} \left[\sup_{0 \leq t \leq T} |M_t| > \kappa \right] \leq \frac{1}{\kappa^p} \mathbb{E} [|M_T|^p] \quad (p \geq 1) \quad (49)$$

for any constant $\kappa > 0$.

In the present context, we are especially interested in the inequality obtained in the case $p = 2$, for which we have the relation

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} M_t^2 \right] \leq 4 \mathbb{E} [M_T^2] \quad (50)$$

which is known as Doob’s L^2 -inequality. Now, setting $M_t = H_t - H_0$ and using equation (46) we obtain

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} (H_t - H_0)^2 \right] \leq 4(V_0 - V_T). \quad (51)$$

In particular, taking the limit $T \rightarrow \infty$, it follows from (41) that

$$\mathbb{E}\left[\sup_{0 \leq t \leq \infty} (H_t - H_0)^2\right] \leq 4V_0 \quad (52)$$

which shows that, on average, the energy stays within two standard deviations of its original value.

This result is consistent with the intuition often arising in physical arguments to the effect that when a system is in a state of uncertain energy, then the energy fluctuates, with a typical fluctuation roughly of the magnitude $\sim \Delta H$. There is no quantum mechanical principle which states that such fluctuations actually occur, but one can see that in a martingale model there may indeed be a natural basis for inferring the existence of fluctuations of the required magnitude. We note that the bound implied by the inequality (52) is independent of the choice of σ , which shows that it is valid also for relatively stable, long-lived states, i.e. those for which $\sigma^2 V_0$ is small.

From (49) we can determine an upper bound on the probability that the magnitude of the energy fluctuation will exceed any designated threshold during the reduction process. Specifically, if we set $p = 2$ and $\kappa = \lambda\sqrt{V_0}$, then taking the limit $T \rightarrow \infty$ we obtain

$$\text{Prob}\left[\sup_{0 \leq t \leq \infty} (H_t - H_0)^2 > \lambda^2 V_0\right] \leq \frac{1}{\lambda^2}. \quad (53)$$

A related bound for the variance process V_t can be obtained by use of Doob's maximal inequality for positive supermartingales (see, e.g., [17], p 58). This relation states that, if X_t is a right-continuous positive supermartingale, then for any constant $k \geq 0$, we have

$$\text{Prob}\left[\sup_{0 \leq t \leq \infty} X_t > k\right] \leq \frac{1}{k} \mathbb{E}[X_0]. \quad (54)$$

In the case of the variance process V_t , which as we have shown is a positive supermartingale, if we set $k = \lambda^2 V_0$, then (54) becomes

$$\text{Prob}\left[\sup_{0 \leq t \leq \infty} V_t > \lambda^2 V_0\right] \leq \frac{1}{\lambda^2}. \quad (55)$$

This relation shows that, during the reduction process, although the energy variance can increase owing to random fluctuations, there is a bound on the probability that the energy uncertainty ever reaches λ times the ensemble average of the initial uncertainty for any given value of λ , and this bound is given by λ^{-2} .

Let us return now to the asymptotic relation $\mathbb{E}[H_\infty] = H_0$ and ask whether the terminal value H_∞ of the energy process actually exists as a random variable. To prove that it does, we make use of the *martingale convergence theorem*, which in a form sufficient for our purpose states that if a continuous martingale M_t satisfies $\mathbb{E}[|M_t|^p] \leq k$ for some $p > 1$ and $k < \infty$, and for all $t \in [0, \infty)$, then there exists a random variable M_∞ satisfying $\mathbb{E}[|M_\infty|^p] \leq k$ and $M_t = \mathbb{E}_t[M_\infty]$, with the properties that $\lim_{t \rightarrow \infty} M_t = M_\infty$ almost surely and that $\lim_{t \rightarrow \infty} \mathbb{E}[|M_t - M_\infty|^p] = 0$.

In the present context, by setting $M_t = H_t - H_0$, we thus deduce the existence of an asymptotic random variable H_∞ with the property that H_t converges to H_∞ almost surely, and $H_t = \mathbb{E}_t[H_\infty]$.

We conclude this section by generalizing (47) to demonstrate that the energy variance process V_t defined by (23) has the natural interpretation

$$V_t = \text{Var}_t[H_\infty]. \quad (56)$$

That is to say, V_t is given by the *conditional variance of the terminal value of the energy, given information up to time t* . To establish this relation we proceed as follows.

For any random variable X on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we define the conditional variance $\text{Var}[X|\mathcal{E}]$ with respect to the σ -subfield $\mathcal{E} \subset \mathcal{F}$ by

$$\text{Var}[X|\mathcal{E}] = \mathbb{E}[(X - \mathbb{E}[X|\mathcal{E}])^2|\mathcal{E}]. \quad (57)$$

It follows as an application of the law of total probability that

$$\text{Var}[X] = \mathbb{E}[\text{Var}[X|\mathcal{E}]] + \text{Var}[\mathbb{E}[X|\mathcal{E}]] \quad (58)$$

the so-called *conditional variance formula*. Thus for example if \mathcal{F}_t ($0 \leq t < \infty$) is a filtration of $(\Omega, \mathcal{F}, \mathbb{P})$ and we write $\text{Var}_t[X] = \text{Var}[X|\mathcal{F}_t]$, then

$$\text{Var}_t[X] = \mathbb{E}_t[(X - \mathbb{E}_t[X])^2] \quad (59)$$

and for the conditional variance formula we have

$$\text{Var}[X] = \mathbb{E}[\text{Var}_t[X]] + \text{Var}[\mathbb{E}_t[X]]. \quad (60)$$

In the problem at hand, we note that in the limit $t \rightarrow \infty$ formula (33) for the variance process V_t takes the form

$$V_0 + \sigma \int_0^\infty \beta_u \, dW_u = \sigma^2 \int_0^\infty V_u^2 \, du. \quad (61)$$

Therefore, taking the conditional expectation of each side of this relation and using formula (25) we deduce that

$$V_0 + \sigma \int_0^t \beta_u \, dW_u = \sigma^2 \mathbb{E}_t \left[\int_0^\infty V_u^2 \, du \right]. \quad (62)$$

Substituting this relation into (33) then gives us

$$\begin{aligned} V_t &= \sigma^2 \mathbb{E}_t \left[\int_0^\infty V_u^2 \, du \right] - \sigma^2 \int_0^t V_u^2 \, du \\ &= \sigma^2 \mathbb{E}_t \left[\int_t^\infty V_u^2 \, du \right] \\ &= \sigma^2 \mathbb{E}_t \left[\left(\int_t^\infty V_u \, dW_u \right)^2 \right] \\ &= \mathbb{E}_t [(H_\infty - H_t)^2] \\ &= \text{Var}_t [H_\infty] \end{aligned} \quad (63)$$

as desired. We note that in the next to last step here we have used (24) together with the conditional Ito isometry

$$\mathbb{E}_t \left[\left(\int_t^T A_u \, dW_u \right)^2 \right] = \mathbb{E}_t \left[\int_t^T A_u^2 \, du \right] \quad (64)$$

valid for any adapted integrand A_u satisfying $\mathbb{E}[\int_0^T A_u^2 \, du] < \infty$.

A positive supermartingale with the property that its expectation goes to zero asymptotically is called a *potential* [18]. The analysis above shows that the variance process associated with quantum state reduction satisfies these conditions and admits a *Doob–Meyer decomposition* of the form

$$V_t = \mathbb{E}_t [Z_\infty] - Z_t \quad (65)$$

where

$$Z_t = \sigma^2 \int_0^t V_u^2 \, du \quad (66)$$

is an increasing process.

6. Reduction probability

The probability π_n of reduction to a specific energy level E_n , under the dynamics governed by the stochastic differential equation (11), can be determined as follows. The method we use is essentially that of [5, 8].

First we observe that for any operator \hat{G} acting on \mathcal{H} , the process G_t for the expectation value of \hat{G} in the state $|\psi_t\rangle$ satisfies

$$dG_t = -i\langle\psi_t|[\hat{G}, \hat{H}]|\psi_t\rangle dt + \frac{1}{4}\sigma^2\langle\psi_t|(\hat{H}\hat{G}\hat{H} - \frac{1}{2}\{\hat{H}^2, \hat{G}\})|\psi_t\rangle dt + \frac{1}{2}\sigma\langle\psi_t|\{(\hat{G} - G_t), (\hat{H} - H_t)\}|\psi_t\rangle dW_t. \quad (67)$$

Here $[\hat{X}, \hat{Y}] = \hat{X}\hat{Y} - \hat{Y}\hat{X}$ and $\{\hat{X}, \hat{Y}\} = \hat{X}\hat{Y} + \hat{Y}\hat{X}$ denote the commutator and the anticommutator, respectively.

The drift term in (67) consists of two parts: the first is the familiar Ehrenfest term involving the commutator with the Hamiltonian; the second is a term of the Lindblad type $[\hat{H}, [\hat{G}, \hat{H}]]$ arising as a consequence of the diffusive dynamics of the state vector. The volatility term in (67), i.e. the coefficient of dW_t , is given by the covariance of \hat{G} and \hat{H} in the state $|\psi_t\rangle$. If \hat{G} and \hat{H} commute, then the drift vanishes, and (67) reduces to

$$dG_t = \sigma(\langle\psi_t|\hat{G}\hat{H}|\psi_t\rangle - G_t H_t) dW_t \quad (68)$$

from which it follows that the process G_t is a martingale [8]. This is consistent with our earlier observation that H_t is itself a martingale, and that the process G_t corresponding to any function of the form $\hat{G} = g(\hat{H})$ is also a martingale.

Now let us consider the projection operator \hat{P}_n for the subspace \mathcal{H}_n of \mathcal{H} spanned by the energy eigenstates with energy E_n . In the case of a nondegenerate eigenvalue, we have $\hat{P}_n = |n\rangle\langle n|$. On the other hand, if E_n is a degenerate eigenvalue, then

$$\hat{P}_n = \sum_{j=1}^{d_n} |n, j\rangle\langle n, j| \quad (69)$$

as in (7), where d_n is the dimension of the subspace \mathcal{H}_n and $|n, j\rangle$ ($j = 1, 2, \dots, d_n$) is an orthonormal basis for \mathcal{H}_n . Clearly \hat{P}_n commutes with the Hamiltonian \hat{H} for any value of n . Furthermore, the relations $\hat{H}\hat{P}_n = \hat{P}_n\hat{H} = E_n\hat{P}_n$ and $\hat{H} = \sum_n E_n\hat{P}_n$ are equivalent on account of the resolution of identity

$$\sum_{n=1}^D \hat{P}_n = 1 \quad (70)$$

where D is the number of distinct energy eigenvalues.

Now let us write

$$P_{nt} = \frac{\langle\psi_t|\hat{P}_n|\psi_t\rangle}{\langle\psi_t|\psi_t\rangle} \quad (71)$$

for the expectation of the projection operator \hat{P}_n in the state $|\psi_t\rangle$. Because \hat{P}_n commutes with the Hamiltonian, we deduce that the process \hat{P}_{nt} is a martingale for each value of n . We note that $\sum_n P_{nt} = 1$ and $\sum_n E_n P_{nt} = H_t$. In particular, by setting $G_t = P_{nt}$, one infers from (68) that

$$dP_{nt} = \sigma P_{nt}(E_n - H_t) dW_t. \quad (72)$$

This stochastic differential equation implies that P_{nt} will continue to fluctuate as long as $H_t \neq E_n$ and $P_{nt} \neq 0$. The solution of (72) is given by $P_{nt} = P_{n0}M_{nt}$, where

$$M_{nt} = \exp\left(\sigma \int_0^t (E_n - H_s) dW_s - \frac{1}{2}\sigma^2 \int_0^t (E_n - H_s)^2 ds\right) \quad (73)$$

and P_{n0} is the initial expectation value of the projection operator \hat{P}_n . This follows from the fact that for any bounded \mathcal{F}_t -adapted process σ_t the solution of the stochastic differential equation $dX_t = \sigma_t X_t dW_t$ ($X_0 > 0$) is

$$X_t = X_0 \exp\left(\int_0^t \sigma_s dW_s - \frac{1}{2} \int_0^t \sigma_s^2 ds\right) \quad (74)$$

which one can verify by an application of Ito's lemma. Because P_{nt} is a martingale, it follows that

$$\mathbb{E}[P_{n\infty}] = P_{n0}. \quad (75)$$

Here $\mathbb{E}[P_{n\infty}]$ is the ensemble average of the expectation value of the projection operator \hat{P}_n at the terminal energy eigenstate of the reduction process. Because $P_{n\infty}$ takes the value one if the terminal energy has eigenvalue E_n and takes the value zero otherwise, it follows that $\mathbb{E}[P_{n\infty}]$ is the *probability of reaching a state with energy E_n* , i.e.

$$\mathbb{E}[P_{n\infty}] = \pi_n. \quad (76)$$

With these observations at hand, we are now in a position to interpret the asymptotic martingale relation (75). If E_n is a nondegenerate eigenvalue, then P_{n0} is the usual expression for the Dirac transition probability from the initial state $|\psi_0\rangle$ to the eigenstate $|n\rangle$, given by

$$P_{n0} = \frac{\langle \psi_0 | n \rangle \langle n | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle \langle n | n \rangle}. \quad (77)$$

Thus we conclude, in the case of a nondegenerate Hamiltonian, that the martingale model for quantum state reduction allows one to *deduce* the correct transition probabilities.

In the case of a degenerate eigenstate, the probability π_n can also be interpreted in terms of a Dirac transition probability. In particular, whether or not the spectrum of the Hamiltonian is degenerate, we can write

$$\begin{aligned} P_{n0} &= \frac{\langle \psi_0 | \hat{P}_n | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle} \\ &= \frac{(\langle \psi_0 | \hat{P}_n | \psi_0 \rangle)^2}{\langle \psi_0 | \psi_0 \rangle \langle \psi_0 | \hat{P}_n | \psi_0 \rangle} \\ &= \frac{\langle P_n \psi_0 | \psi_0 \rangle \langle \psi_0 | P_n \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle \langle \psi_0 | \hat{P}_n | \psi_0 \rangle} \\ &= \frac{\langle P_n \psi_0 | \psi_0 \rangle \langle \psi_0 | P_n \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle \langle \psi_0 | \hat{P}_n^2 | \psi_0 \rangle} \\ &= \frac{\langle P_n \psi_0 | \psi_0 \rangle \langle \psi_0 | P_n \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle \langle P_n \psi_0 | P_n \psi_0 \rangle} \end{aligned} \quad (78)$$

where the Lüders state $|P_n \psi\rangle$ is defined by

$$|P_n \psi_0\rangle \triangleq \hat{P}_n |\psi_0\rangle. \quad (79)$$

Therefore, by virtue of (75) and (78), we see that *the probability of obtaining the eigenvalue E_n is given by the Dirac transition probability from the given initial state $|\psi_0\rangle$ to the Lüders state $|P_n \psi_0\rangle$.*

The interesting point here is that, once again, while this is an *assumption* in standard quantum theory, it arises as a *deduction* in the martingale model for quantum state reduction.

In fact, we can demonstrate, in the case of a degenerate eigenvalue, that the reduction necessarily results in the Lüders state if the corresponding eigenvalue is obtained. This can be seen as follows.

For each value of $n = 1, 2, \dots, D$, such that $\hat{P}_n|\psi_0\rangle \neq 0$ let us write

$$|n, 1\rangle = \frac{\hat{P}_n|\psi_0\rangle}{\langle\psi_0|\hat{P}_n|\psi_0\rangle^{1/2}} \quad (80)$$

for a basis vector corresponding to the normalized Lüders state for that projection operator, and let $|n, j\rangle$, $j \neq 1$, be an associated basis for the states orthogonal to $|n, 1\rangle$ that lie in the subspace \mathcal{H}_n spanned by eigenstates of energy E_n . The operator

$$\begin{aligned} \hat{\Pi}_n &\triangleq \sum_{j=2}^{d_n} |n, j\rangle\langle n, j| \\ &= \hat{P}_n - |n, 1\rangle\langle n, 1| \end{aligned} \quad (81)$$

thus projects onto the subspace of \mathcal{H}_n consisting of vectors *orthogonal* to the Lüders state for that value of n . Evidently, we have

$$\hat{\Pi}_n|\psi_0\rangle = 0 \quad (82)$$

which follows from (80) and the fact that $\hat{\Pi}_n\hat{P}_n = \hat{\Pi}_n$. Since the projection operator $\hat{\Pi}_n$ commutes with the Hamiltonian, the process

$$\Pi_{nt} = \frac{\langle\psi_t|\hat{\Pi}_n|\psi_t\rangle}{\langle\psi_t|\psi_t\rangle} \quad (83)$$

is a martingale, the initial value of which is $\Pi_{n0} = 0$ on account of the relation (82). Therefore, by virtue of the martingale relation $\mathbb{E}[\Pi_{n\infty}] = \Pi_{n0}$, we deduce that $\mathbb{E}[\Pi_{n\infty}] = 0$. Now, $\Pi_{n\infty}$ is a non-negative random variable. Therefore, if $\mathbb{E}[\Pi_{n\infty}] = 0$ then $\Pi_{n\infty} = 0$ almost surely. It follows that the terminal state must be orthogonal to the subspace of \mathcal{H}_n spanned by states with energy E_n that are orthogonal to the Lüders state. As a consequence, we see that *if reduction occurs to a state of energy E_n , then that state must be the Lüders state corresponding to that eigenvalue.*

In fact, we deduce a stronger result: namely, that *the stochastic motion of the state vector, during the course of the reduction process, is necessarily confined to the D -dimensional subspace of \mathcal{H} spanned by the Lüders states $\hat{P}_n|\psi_0\rangle$, for $n = 1, 2, \dots, D$, where D is the number of distinct energy eigenvalues and \hat{P}_n is the projection operator onto the subspace \mathcal{H}_n of \mathcal{H} spanned by eigenstates with eigenvalue E_n .*

The proof of this theorem follows from the fact that, for each n , the process Π_{nt} is a martingale, and because $\Pi_{n0} = 0$ we have $\mathbb{E}[\Pi_{nt}] = 0$ for all $t \geq 0$ and thus $\Pi_{nt} = 0$ for all $t \geq 0$. Therefore the state vector $|\psi_t\rangle$ always lies in the space spanned by the vectors $\hat{P}_n|\psi_0\rangle$ for $n = 1, 2, \dots, D$.

A similar analysis is valid in the more general situation for which the dynamics of $|\psi_t\rangle$ are given by a stochastic differential equation of the form

$$d|\psi_t\rangle = -i\hat{H}|\psi_t\rangle dt - \frac{1}{8} \sum_{\alpha=1}^r \sigma_\alpha^2 (\hat{F}_\alpha - F_{\alpha t})^2 |\psi_t\rangle dt + \frac{1}{2} \sum_{\alpha=1}^r \sigma_\alpha (\hat{F}_\alpha - F_{\alpha t}) dW_t^\alpha. \quad (84)$$

Here \hat{F}_α ($\alpha = 1, 2, \dots, r$) represents a commuting family of observables, each of which also commutes with the Hamiltonian \hat{H} , the σ_α are associated coupling constants, and we write $F_{\alpha t}$ for the expectation of \hat{F}_α in the state $|\psi_t\rangle$. In this case W_t^α denotes a standard r -dimensional Brownian motion, and the reduction proceeds to a common eigenstate of operators \hat{F}_α ($\alpha = 1, 2, \dots, r$). Most of the results of this paper are applicable *mutatis mutandis* to this more general class of reduction process, though in what follows we shall, for simplicity, continue to confine the detailed discussion to the case of the energy-based reduction (11).

7. The case of an initially mixed state

Thus far we have considered the role of the Lüders postulate as it applies to an initially *pure* state $|\psi_0\rangle$, and we have demonstrated that the postulate follows directly as a consequence of the martingale model for quantum state reduction. The Lüders postulate is, however, applicable in a more general context as well: namely, when the initial state is specified as a mixture with density matrix $\hat{\rho}_0$. In that case, when an observable F is measured, the associated state reduction is given by the Lüders rule

$$\hat{\rho}_0 \mapsto \frac{\hat{P}_n \hat{\rho}_0 \hat{P}_n}{\text{Tr} \hat{P}_n \hat{\rho}_0} \quad (85)$$

for the density matrix, if the measurement result is the eigenvalue f_n , and this occurs with probability

$$\pi_n = \text{Tr} \hat{P}_n \hat{\rho}_0. \quad (86)$$

Here, as before, \hat{P}_n denotes the projection operator onto the subspace \mathcal{H}_n of \mathcal{H} spanned by eigenstates with the eigenvalue f_n , which may or may not be degenerate.

The interpretation of an expression involving density matrices, such as (85), is best understood in terms of ensemble averages. Thus (85) means that if initially $\hat{\rho}_0$ can be used to compute the expectation of any observable G , not necessarily compatible with F , then after F is measured, and if the result f_n is observed, the density matrix $\hat{P}_n \hat{\rho}_0 \hat{P}_n / \text{Tr} \hat{P}_n \hat{\rho}_0$ can be used to compute the expectation of G in a subsequent measurement.

Additionally, given the initial density matrix $\hat{\rho}_0$, if F is measured but no note is taken of the result, then the ensemble average for a subsequent measurement of the observable G is $\text{Tr} \hat{\rho}_\infty \hat{G}$, where

$$\hat{\rho}_\infty = \sum_n \hat{P}_n \hat{\rho}_0 \hat{P}_n. \quad (87)$$

It should be borne in mind that these transformations, while generally regarded as part of the standard apparatus of quantum theory, are not derivable from any of the more basic assumptions of quantum mechanics, and have to be regarded as constituting an additional postulate. See, e.g., [19] for an illuminating brief account of the status of the projection postulate in quantum mechanics, and its relation to state reduction. It is interesting to note that von Neumann, in his original splendid work on the subject [1], apparently failed to offer a satisfactory expression for the density matrix in the case of the measurement of an observable with a degenerate spectrum, a deficiency only later rectified by Lüders and others [2] (cf [20], section 9, and the remark attributed to Wightman on p 550 of [21]).

The general Lüders rule (85) has the important property that, in the measurement of an observable with a degenerate spectrum, if the initial state is not pure, then the final state need not be pure, if the result of the measurement is one of the degenerate eigenvalues.

Now let us see if we can gain a clearer understanding of the general Lüders formulae (85)–(87) by consideration of the martingale model for quantum state reduction. In the theory of stochastic differential equations, it is acceptable that the initial value of the random process should itself be a random variable; thus it is merely a special case when $|\psi_0\rangle$ in (11) is known. The deterministic case corresponds to the situation where the initial density matrix is pure, i.e. of rank one. In the general case, where $|\psi_0\rangle$ is random, i.e. given by a mixture, the corresponding initial density matrix $\hat{\rho}_0$ is the ensemble average

$$\hat{\rho}_0 = \mathbb{E} [|\Psi_0\rangle\langle\Psi_0|] \quad (88)$$

where $|\Psi_0\rangle$ is a random initial state vector, which we assume to be normalized. Then for the final density matrix we have

$$\hat{\rho}_\infty = \sum_n \mathbb{E} \left[\pi_n(\Psi_0) \frac{\hat{P}_n |\Psi_0\rangle \langle \Psi_0| \hat{P}_n}{\langle \Psi_0 | \hat{P}_n | \Psi_0 \rangle} \right] \quad (89)$$

as a consequence of the reduction

$$|\Psi_0\rangle \mapsto \frac{\hat{P}_n |\Psi_0\rangle}{\langle \Psi_0 | \hat{P}_n | \Psi_0 \rangle^{1/2}} \quad (90)$$

where $\pi_n(\Psi_0)$ is the conditional probability that the eigenvalue E_n is obtained, given the random initial state $|\Psi_0\rangle$. However, this conditional probability is given by

$$\pi_n(\Psi_0) = \langle \Psi_0 | \hat{P}_n | \Psi_0 \rangle \quad (91)$$

i.e. the Dirac transition probability to the random Lüders state determined by the random initial state $|\Psi_0\rangle$, in accordance with (78). As a consequence we see that (89) simplifies to give

$$\begin{aligned} \hat{\rho}_\infty &= \sum_n \mathbb{E} [\hat{P}_n |\Psi_0\rangle \langle \Psi_0| \hat{P}_n] \\ &= \sum_n \hat{P}_n \mathbb{E} [|\Psi_0\rangle \langle \Psi_0|] \hat{P}_n \\ &= \sum_n \hat{P}_n \hat{\rho}_0 \hat{P}_n \end{aligned} \quad (92)$$

and thus we obtain (87). Additionally we have

$$\hat{\rho}_\infty = \sum_n \pi_n \hat{\rho}_{n\infty} \quad (93)$$

where

$$\hat{\rho}_{n\infty} = \frac{\hat{P}_n \hat{\rho}_0 \hat{P}_n}{\text{Tr} \hat{P}_n \hat{\rho}_0} \quad (94)$$

is the reduced or ‘conditional’ density matrix, given that the observer has knowledge of the result $H = E_n$, and

$$\begin{aligned} \pi_n &= \mathbb{E} [\pi_n(\Psi_0)] \\ &= \mathbb{E} [\langle \Psi_0 | \hat{P}_n | \Psi_0 \rangle] \\ &= \mathbb{E} [\text{Tr} \hat{P}_n |\Psi_0\rangle \langle \Psi_0|] \\ &= \text{Tr} \hat{P}_n \mathbb{E} [|\Psi_0\rangle \langle \Psi_0|] \\ &= \text{Tr} \hat{P}_n \hat{\rho}_0 \end{aligned} \quad (95)$$

is the probability of this result.

8. Dynamics of the density matrix

To gain further insight into the case where the initial density matrix is not pure, we can make a computation of the dynamics for $\hat{\rho}_t$. This can be achieved by examination of the Lindblad equation associated with the stochastic differential equation (11), which in this case turns out to be solvable.

If we start with equation (67) for the dynamics of the expectation value of an arbitrary operator \hat{G} in the random state $|\psi_t\rangle$, and take the ensemble average

$$\mathbb{E} [G_t] = \text{Tr} \hat{G} \hat{\rho}_t \quad (96)$$

where $\hat{\rho}_t = \mathbb{E}[|\psi_t\rangle\langle\psi_t|]$, we find that

$$\begin{aligned} d\mathbb{E}[G_t] &= \text{Tr}\hat{G} d\hat{\rho}_t \\ &= -i\text{Tr}\hat{\rho}_t[\hat{G}, \hat{H}] dt + \frac{1}{4}\sigma^2\text{Tr}\hat{\rho}_t(\hat{H}\hat{G}\hat{H} - \frac{1}{2}\hat{H}^2\hat{G} - \frac{1}{2}\hat{G}\hat{H}^2) dt \\ &= -i\text{Tr}\hat{G}[\hat{H}, \hat{\rho}_t] dt + \frac{1}{4}\sigma^2\text{Tr}\hat{G}(\hat{H}\hat{\rho}_t\hat{H} - \frac{1}{2}\hat{\rho}_t\hat{H}^2 - \frac{1}{2}\hat{H}^2\hat{\rho}_t) dt \end{aligned} \quad (97)$$

where in the second equality we make use of the cyclic property of the trace operator. This relation has to hold for any observable G , from which it follows that

$$\partial_t\hat{\rho}_t = -i[\hat{H}, \hat{\rho}_t] + \frac{1}{4}\sigma^2(\hat{H}\hat{\rho}_t\hat{H} - \frac{1}{2}\hat{H}^2\hat{\rho}_t - \frac{1}{2}\hat{\rho}_t\hat{H}^2) \quad (98)$$

where $\partial_t = \partial/\partial t$. This is the general equation of the Lindblad type [22, 23] associated with the stochastic differential equation (11), for which $\frac{1}{2}\sigma\hat{H}$ is the corresponding Lindblad operator.

Now we consider the problem of solving the Lindblad equation (98) subject to an arbitrary specification of the initial density matrix $\hat{\rho}_0$. For convenience we switch to a Heisenberg representation in which the density matrix is defined by the operator

$$\hat{r}_t \triangleq e^{i\hat{H}t}\hat{\rho}_te^{-i\hat{H}t}. \quad (99)$$

This has the effect of removing the purely unitary part of the evolution. For the dynamics of \hat{r}_t we have

$$\partial_t\hat{r}_t = e^{i\hat{H}t}(\partial_t\hat{\rho}_t)e^{-i\hat{H}t} + i[\hat{H}, \hat{r}_t] \quad (100)$$

and therefore

$$\partial_t\hat{r}_t = \frac{1}{4}\sigma^2(\hat{H}\hat{r}_t\hat{H} - \frac{1}{2}\hat{H}^2\hat{r}_t - \frac{1}{2}\hat{r}_t\hat{H}^2). \quad (101)$$

Let us write \hat{P}_n for the projection operator onto the subspace \mathcal{H}_n . Then, because $\hat{P}_n\hat{H} = \hat{H}\hat{P}_n = E_n\hat{P}_n$, if we multiply each side of equation (101) by \hat{P}_n on both the right and the left we obtain

$$\partial_t(\hat{P}_n\hat{r}_t\hat{P}_n) = 0 \quad (102)$$

from which it follows that $\hat{P}_n\hat{r}_t\hat{P}_n$ is a constant of the motion. In particular, we have $\hat{P}_n\hat{r}_0\hat{P}_n = \hat{P}_n\hat{r}_\infty\hat{P}_n$, and thus $\hat{P}_n\hat{\rho}_0\hat{P}_n = \hat{P}_n\hat{\rho}_\infty\hat{P}_n$, and therefore

$$\sum_n \hat{P}_n\hat{\rho}_0\hat{P}_n = \sum_n \hat{P}_n\hat{\rho}_\infty\hat{P}_n. \quad (103)$$

Because the terminal state is necessarily a mixture of energy eigenstates we have

$$\sum_n \hat{P}_n\hat{\rho}_\infty\hat{P}_n = \hat{\rho}_\infty \quad (104)$$

from which by use of (103) we immediately infer the general form of the Lüders reduction postulate (87).

To proceed further we define the operator matrix valued process $\hat{\mathcal{R}}_{nm}(t)$ by

$$\hat{\mathcal{R}}_{nm}(t) \triangleq \hat{P}_n\hat{r}_t\hat{P}_m. \quad (105)$$

For each of the values of n and m , $\hat{\mathcal{R}}_{nm}(t)$ is a time-dependent Hermitian operator. Here, $n, m = 1, 2, \dots, D$, where D is the number of distinct energy levels. From equation (101) for the dynamics of \hat{r}_t we deduce, by use of the relation $\hat{P}_n\hat{H} = E_n\hat{P}_n$, that

$$\partial_t\hat{\mathcal{R}}_{nm}(t) = -\frac{1}{8}\sigma^2(E_n - E_m)^2\hat{\mathcal{R}}_{nm}(t). \quad (106)$$

The general solution of the ordinary differential equation (106) is given by

$$\begin{aligned} \hat{\mathcal{R}}_{nm}(t) &= \hat{\mathcal{R}}_{nm}(0) \exp\left(-\frac{1}{8}\sigma^2(E_n - E_m)^2t\right) \\ &= \hat{P}_n\hat{r}_0\hat{P}_m \exp\left(-\frac{1}{8}\sigma^2(E_n - E_m)^2t\right). \end{aligned} \quad (107)$$

On the other hand, by use of the resolution of the identity (70) it follows from (105) that

$$\begin{aligned}\hat{r}_t &= \sum_{n,m} \hat{\mathcal{R}}_{nm}(t) \\ &= \sum_{n,m} \hat{P}_n \hat{r}_0 \hat{P}_m \exp\left(-\frac{1}{8}\sigma^2(E_n - E_m)^2 t\right) \\ &= \sum_{n \neq m} \hat{P}_n \hat{r}_0 \hat{P}_m \exp\left(-\frac{1}{8}\sigma^2(E_n - E_m)^2 t\right) + \sum_n \hat{P}_n \hat{r}_0 \hat{P}_n.\end{aligned}\quad (108)$$

Therefore, by inverting the transformation (99), we obtain the solution of the Lindblad equation in the original Schrödinger picture as:

$$\hat{\rho}_t = \sum_{n \neq m} \hat{P}_n \hat{\rho}_0 \hat{P}_m \exp\left(-i(E_n - E_m)t - \frac{1}{8}\sigma^2(E_n - E_m)^2 t\right) + \sum_n \hat{P}_n \hat{\rho}_0 \hat{P}_n.\quad (109)$$

We recover the initial state $\hat{\rho}_0$ by setting $t = 0$ in the right-hand side of (109). The off-diagonal terms are damped away exponentially at the rate $\frac{1}{8}\sigma^2 V_{nm}$ as $t \rightarrow \infty$, where $V_{nm} = (E_n - E_m)^2$ is the square of the spread between relevant energy levels, and we are left with the Lüders state (87) for the terminal density matrix $\hat{\rho}_\infty$.

It is interesting to observe that the Lüders state, obtained by the limit as $t \rightarrow \infty$ of the density matrix associated with the reduction process (11), coincides with the asymptotic time average of the density matrix in the case of a purely unitary evolution governed by the von Neumann equation

$$\frac{\partial \hat{\rho}_t}{\partial t} = -i[\hat{H}, \hat{\rho}_t]\quad (110)$$

for which the solution is $\hat{\rho}_t = e^{-i\hat{H}t} \hat{\rho}_0 e^{i\hat{H}t}$. More specifically, if we write

$$\langle \hat{\rho} \rangle_T \triangleq \frac{1}{T} \int_0^T \hat{\rho}_t dt\quad (111)$$

for the time average of $\hat{\rho}_t$ up to time T , then we find that

$$\lim_{T \rightarrow \infty} \langle \hat{\rho} \rangle_T = \sum_n \hat{P}_n \hat{\rho}_0 \hat{P}_n\quad (112)$$

where \hat{P}_n is the projection operator onto the subspace $\mathcal{H}_n \subset \mathcal{H}$ spanned by the states of energy E_n . This result can be verified directly by use of the resolution of the identity (70). The calculation is as follows:

$$\begin{aligned}\langle \hat{\rho} \rangle_T &= \frac{1}{T} \sum_{m,n} \int_0^T \hat{P}_n e^{-i\hat{H}t} \hat{\rho}_0 e^{i\hat{H}t} \hat{P}_m dt \\ &= \frac{1}{T} \sum_{m,n} \hat{P}_n \hat{\rho}_0 \hat{P}_m \int_0^T e^{-i(E_n - E_m)t} dt \\ &= \sum_n \hat{P}_n \hat{\rho}_0 \hat{P}_n + \frac{1}{T} \sum_{m \neq n} \hat{P}_n \hat{\rho}_0 \hat{P}_m \left(\frac{\sin(\omega_{nm}T)}{\omega_{nm}} + i \frac{\cos(\omega_{nm}T) - 1}{\omega_{nm}} \right)\end{aligned}\quad (113)$$

where $\omega_{nm} = E_n - E_m$. Therefore, in the limit $T \rightarrow \infty$ the off-diagonal terms drop out, and we recover (112). For a closely related result see [24].

9. Change of measure

We return now to the stochastic differential equation (11) governing quantum state reduction with a view to gaining further insights into the nature of the resulting dynamics. We shall demonstrate in this section how a ‘change of measure’ technique can be used to solve (11) and thus, in effect, to construct an explicit unravelling of the Lindblad equation (98). The change of measure method has been found to be useful in other applications of stochastic calculus as well e.g. mathematical finance. The general problem of formulating an appropriate unravelling of the Lindblad equation in a given physical context is a matter of considerable interest in a number of areas of modern physics [25–30].

We begin with the following remark. Let $\hat{\mu}_t$ and $\hat{\sigma}_t$ be bounded \mathcal{F}_t -adapted operator-valued processes on $(\Omega, \mathcal{F}, \mathbb{P})$ with the property that for all $s, t \in [0, \infty)$ the random matrices $\hat{\mu}_s, \hat{\mu}_t, \hat{\sigma}_s$ and $\hat{\sigma}_t$ mutually commute. Then the stochastic differential equation

$$d|\psi_t\rangle = \hat{\mu}_t|\psi_t\rangle dt + \hat{\sigma}_t|\psi_t\rangle dW_t \quad (114)$$

has the unique solution

$$|\psi_t\rangle = \exp\left(\int_0^t (\hat{\mu}_s - \frac{1}{2}\hat{\sigma}_s^2) ds + \int_0^t \hat{\sigma}_s dW_s\right) |\psi_0\rangle. \quad (115)$$

Here we allow for the possibility that the initial state $|\psi_0\rangle$ may be random. A straightforward application of Ito’s lemma shows that (115) leads back to (114). In the case of the reduction process (11), which is evidently of the form (114), we can write

$$\hat{\mu}_t = -i\hat{H} - \frac{1}{8}\sigma^2(\hat{H} - H_t)^2 \quad \hat{\sigma}_t = \frac{1}{2}\sigma(\hat{H} - H_t). \quad (116)$$

It follows therefore that

$$|\psi_t\rangle = \exp\left(-i\hat{H}t - \frac{1}{4}\sigma^2 \int_0^t (\hat{H} - H_s)^2 ds + \frac{1}{2}\sigma \int_0^t (\hat{H} - H_s) dW_s\right) |\psi_0\rangle. \quad (117)$$

This is still an implicit solution for $|\psi_t\rangle$, because $H_s = \langle \psi_s | \hat{H} | \psi_s \rangle$. Nevertheless, as a consequence of (117) we see that the evolution of the state vector according to (11) can be expressed in the simple form

$$|\psi_t\rangle = \hat{U}_t \hat{R}_t |\psi_0\rangle \quad (118)$$

where the operator-valued process \hat{U}_t is defined by

$$\hat{U}_t \triangleq \exp(-i\hat{H}t) \quad (119)$$

and the operator-valued process \hat{R}_t is defined by

$$\hat{R}_t \triangleq \exp\left(\frac{1}{2}\sigma \int_0^t (\hat{H} - H_s) dW_s - \frac{1}{4}\sigma^2 \int_0^t (\hat{H} - H_s)^2 ds\right). \quad (120)$$

We note that \hat{U}_t is unitary and that \hat{U}_t and \hat{R}_t commute. The square of \hat{R}_t , which we denote by \hat{M}_t , is an operator-valued martingale. The fact that \hat{M}_t satisfies the martingale condition $\mathbb{E}_s[\hat{M}_t] = \hat{M}_s$ is evident from the expression

$$\hat{M}_t = \exp\left(\sigma \int_0^t (\hat{H} - H_s) dW_s - \frac{1}{2}\sigma^2 \int_0^t (\hat{H} - H_s)^2 ds\right). \quad (121)$$

In particular, if \hat{P}_n is the projection operator onto the subspace $\mathcal{H}_n \subset \mathcal{H}$ spanned by the states of energy E_n , then we find that

$$\hat{M}_t = \sum_n \hat{P}_n M_{nt} \quad (122)$$

where M_{nt} is given by (73). We note that for each value of n the process M_{nt} is in fact an exponential martingale.

Now suppose that \hat{G} is an observable that commutes with the Hamiltonian \hat{H} . Then for its expectation in the state $|\psi_t\rangle$ we have

$$\begin{aligned} G_t &= \langle \psi_t | \hat{G} | \psi_t \rangle \\ &= \langle \psi_0 | \hat{R}_t \hat{U}_t^\dagger \hat{G} \hat{U}_t \hat{R}_t | \psi_0 \rangle \\ &= \langle \psi_0 | \hat{G} \hat{M}_t | \psi_0 \rangle \end{aligned} \quad (123)$$

and therefore

$$\begin{aligned} \mathbb{E}_s [G_t] &= \mathbb{E}_s [\langle \psi_0 | \hat{G} \hat{M}_t | \psi_0 \rangle] \\ &= \langle \psi_0 | \hat{G} \mathbb{E}_s [\hat{M}_t] | \psi_0 \rangle \\ &= \langle \psi_0 | \hat{G} \hat{M}_s | \psi_0 \rangle \\ &= \langle \psi_0 | \hat{R}_s \hat{G} \hat{R}_s | \psi_0 \rangle \\ &= \langle \psi_0 | \hat{R}_s \hat{U}_s^\dagger \hat{G} \hat{U}_s \hat{R}_s | \psi_0 \rangle \\ &= \langle \psi_s | \hat{G} | \psi_s \rangle \end{aligned} \quad (124)$$

which shows that G_t is a martingale. In this way we are able to verify directly that the dynamical law (11) implies that the expectation value of any observable that commutes with the Hamiltonian is a weakly conserved quantity.

To proceed further we note that it is a straightforward algebraic exercise to verify that \hat{M}_t can be expressed as the following quotient:

$$\hat{M}_t = \frac{\exp(\sigma \int_0^t \hat{H} (dW_s + \sigma H_s ds) - \frac{1}{2} \sigma^2 \int_0^t \hat{H}^2 ds)}{\exp(\sigma \int_0^t H_s (dW_s + \sigma H_s ds) - \frac{1}{2} \sigma^2 \int_0^t H_s^2 ds)}. \quad (125)$$

In particular, let us define the ‘modified’ Brownian motion W_t^* by

$$W_t^* \triangleq W_t + \sigma \int_0^t H_s ds \quad (126)$$

so $dW_t^* = dW_t + \sigma H_t dt$. Then, because \hat{H} is constant, we can write \hat{M}_t in the simple form

$$\hat{M}_t = \frac{1}{\Lambda_t^*} \exp\left(\sigma \hat{H} W_t^* - \frac{1}{2} \sigma^2 \hat{H}^2 t\right) \quad (127)$$

where

$$\Lambda_t^* \triangleq \exp\left(\sigma \int_0^t H_s dW_s^* - \frac{1}{2} \sigma^2 \int_0^t H_s^2 ds\right). \quad (128)$$

The significance of the processes W_t^* and Λ_t^* will become apparent shortly.

We have already verified that (11) preserves the norm of $|\psi_0\rangle$. If we assume that $\langle \psi_0 | \psi_0 \rangle = 1$, then it follows from (118) that $\langle \psi_0 | \hat{M}_t | \psi_0 \rangle = 1$ for all t . Thus we deduce from (125) and (126) that

$$\Lambda_t^* = \langle \psi_0 | \exp(\sigma \hat{H} W_t^* - \frac{1}{2} \sigma^2 \hat{H}^2 t) | \psi_0 \rangle. \quad (129)$$

As a consequence we can write

$$\hat{M}_t = \frac{\exp(\sigma \hat{H} W_t^* - \frac{1}{2} \sigma^2 \hat{H}^2 t)}{\langle \psi_0 | \exp(\sigma \hat{H} W_t^* - \frac{1}{2} \sigma^2 \hat{H}^2 t) | \psi_0 \rangle} \quad (130)$$

which has the important effect of localizing the dependence of \hat{M}_t on H_t in the modified Brownian motion W_t^* . The process H_t in turn is given by (12), from which it follows that $H_t = \langle \psi_0 | \hat{H} \hat{M}_t | \psi_0 \rangle$. Therefore, by use of (130) we have

$$H_t = \frac{\langle \psi_0 | \hat{H} \exp(\sigma \hat{H} W_t^* - \frac{1}{2} \sigma^2 \hat{H}^2 t) | \psi_0 \rangle}{\langle \psi_0 | \exp(\sigma \hat{H} W_t^* - \frac{1}{2} \sigma^2 \hat{H}^2 t) | \psi_0 \rangle} \tag{131}$$

which shows that H_t can be expressed as a function of W_t^* and t . This is given explicitly by

$$H_t = \frac{\sum_n \pi_n E_n \exp(\sigma E_n W_t^* - \frac{1}{2} \sigma^2 E_n^2 t)}{\sum_n \pi_n \exp(\sigma E_n W_t^* - \frac{1}{2} \sigma^2 E_n^2 t)} \tag{132}$$

where as usual π_n denotes the probability that the eigenvalue attained is E_n , given the initial state $|\psi_0\rangle$. We also note that

$$\Lambda_t^* = \sum_n \pi_n \exp(\sigma E_n W_t^* - \frac{1}{2} \sigma^2 E_n^2 t) \tag{133}$$

and that

$$\hat{M}_t = \frac{\sum_n \hat{P}_n \exp(\sigma E_n W_t^* - \frac{1}{2} \sigma^2 E_n^2 t)}{\sum_n \pi_n \exp(\sigma E_n W_t^* - \frac{1}{2} \sigma^2 E_n^2 t)}. \tag{134}$$

Now we proceed to examine the processes W_t^* and Λ_t^* more closely. This is the point at which we introduce the highly useful concept of a *change of probability measure*. We shall see in what follows that there exists a change of measure $\mathbb{P} \rightarrow \mathbb{Q}$ such that for any given finite interval of time $[0, T]$ the process W_t^* for $t \in [0, T]$ is a Brownian motion with the probability space $(\Omega, \mathcal{F}_T, \mathbb{Q})$ on the filtration \mathcal{F}_t ($0 \leq t \leq T$). The implication of this is that with respect to the measure \mathbb{Q} the basic processes H_t , Λ_t^* , and \hat{M}_t can be expressed in terms of ratios of sums of geometric Brownian motions, thus offering a significant element of analytic tractability.

We begin with a few mathematical preliminaries concerning the change of measure technique. Given the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we recall that 1_A denotes the indicator function of the event $A \in \mathcal{F}$. Thus for each $\omega \in \Omega$ we have $1_A(\omega) = 1$ if $\omega \in A$ and $1_A(\omega) = 0$ if $\omega \notin A$. It follows that

$$\text{Prob}^{\mathbb{P}}[A] = \mathbb{E}^{\mathbb{P}}[1_A] \tag{135}$$

where $\text{Prob}^{\mathbb{P}}$ and $\mathbb{E}^{\mathbb{P}}$ denote probability and expectation with respect to the measure \mathbb{P} .

Now let Λ be a positive random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then we can define a new probability measure \mathbb{Q} on the underlying measurable space (Ω, \mathcal{F}) by the formula

$$\text{Prob}^{\mathbb{Q}}[A] = \frac{\mathbb{E}^{\mathbb{P}}[\Lambda 1_A]}{\mathbb{E}^{\mathbb{P}}[\Lambda]}. \tag{136}$$

Because Λ is positive, this relation is invertible and we have

$$\text{Prob}^{\mathbb{P}}[A] = \frac{\mathbb{E}^{\mathbb{Q}}[\Lambda^* 1_A]}{\mathbb{E}^{\mathbb{Q}}[\Lambda^*]} \tag{137}$$

where $\Lambda^* = 1/\Lambda$. The two probability measures \mathbb{P} and \mathbb{Q} in this case are said to be *equivalent* in the sense that they agree on null sets, i.e. for all $A \in \mathcal{F}$ we have $\text{Prob}^{\mathbb{P}}[A] = 0$ if and only if $\text{Prob}^{\mathbb{Q}}[A] = 0$.

In the case of a filtered probability space some important additional structure arises in this connection. Suppose the process Λ_t is a positive martingale on $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to the filtration \mathcal{F}_t , satisfying $\Lambda_0 = 1$. For any fixed value of t the random variable Λ_t can be used to

define a measure \mathbb{Q}_t on (Ω, \mathcal{F}_t) according to the procedure outlined in the previous paragraph. It follows then from (136) by virtue of the martingale property of Λ_t that

$$\text{Prob}^{\mathbb{Q}_t}[A] = \mathbb{E}^{\mathbb{P}}[\Lambda_t 1_A] \quad (138)$$

for all $A \in \mathcal{F}_t$. We note that if $s \leq t$ then $\text{Prob}^{\mathbb{Q}_s}[A] = \text{Prob}^{\mathbb{Q}_t}[A]$ for all $A \in \mathcal{F}_s$. This is because

$$\begin{aligned} \text{Prob}^{\mathbb{Q}_t}[A] &= \mathbb{E}^{\mathbb{P}}[\Lambda_t 1_A] \\ &= \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[\Lambda_t 1_A | \mathcal{F}_s]] \\ &= \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[\Lambda_t | \mathcal{F}_s] 1_A] \\ &= \mathbb{E}^{\mathbb{P}}[\Lambda_s 1_A] \\ &= \text{Prob}^{\mathbb{Q}_s}[A]. \end{aligned} \quad (139)$$

Therefore, for any finite interval of time $[0, T]$ the measure thus obtained on (Ω, \mathcal{F}_T) is independent of the specific choice of T . Thus we can drop the suffix on \mathbb{Q} and speak of the change of measure $\mathbb{P} \rightarrow \mathbb{Q}$ induced by the given density martingale Λ_t .

The key result making use of this apparatus that we require in what follows is the theorem of Girsanov (see, e.g., [31]). Let $[0, T]$ be a fixed interval of time, and W_t a Brownian motion on (Ω, \mathcal{F}_T) with respect to the filtration \mathcal{F}_t ($0 \leq t \leq T$) and the measure \mathbb{P} . Suppose that the process λ_t is \mathcal{F}_t -adapted and that

$$\Lambda_t = \exp\left(-\int_0^t \lambda_s dW_s - \frac{1}{2} \int_0^t \lambda_s^2 ds\right) \quad (140)$$

is a martingale. Then Girsanov's theorem states that the modified process

$$W_t^* \triangleq W_t + \int_0^t \lambda_s ds \quad (141)$$

is a Brownian motion with respect to the equivalent measure \mathbb{Q} induced by the density martingale Λ_t . For Λ_t to be a martingale it suffices that λ_s should satisfy the Novikov condition

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2} \int_0^T \lambda_s^2 ds\right)\right] < \infty. \quad (142)$$

In particular, if λ_t is bounded, then Λ_t is a martingale.

If Λ_t is a \mathbb{P} -martingale then the associated process $\Lambda_t^* = 1/\Lambda_t$ given by

$$\Lambda_t^* = \exp\left(+\int_0^t \lambda_s dW_s^* - \frac{1}{2} \int_0^t \lambda_s^2 ds\right) \quad (143)$$

is a \mathbb{Q} -martingale, and induces the inverse change of measure $\mathbb{Q} \rightarrow \mathbb{P}$. In particular, for any \mathcal{F}_t -measurable random variable X_t we have the following formulae for the calculation of expectations:

$$\mathbb{E}^{\mathbb{P}}[X_t] = \frac{1}{\Lambda_s^*} \mathbb{E}_s^{\mathbb{Q}}[\Lambda_t^* X_t] \quad (144)$$

and its reversal

$$\mathbb{E}_s^{\mathbb{Q}}[X_t] = \frac{1}{\Lambda_s} \mathbb{E}_s^{\mathbb{P}}[\Lambda_t X_t]. \quad (145)$$

Returning to the matter at hand, we note that for quantum state reduction the process λ_t is given by σH_t , which satisfies the Novikov condition because H_t is bounded. The corresponding change of measure density martingale is given by

$$\Lambda_t = \exp\left(-\sigma \int_0^t H_s dW_s - \frac{1}{2} \sigma^2 \int_0^t H_s^2 ds\right) \quad (146)$$

for the transformation $\mathbb{P} \rightarrow \mathbb{Q}$. The process W_t^* as defined by (126) is a \mathbb{Q} -Brownian motion. The associated inverse transformation $\mathbb{Q} \rightarrow \mathbb{P}$ is induced by the process Λ_t^* defined in (128).

Now we are in a position to give a complete characterization of the solution of the dynamical equation (11) for the state reduction problem valid over any finite time interval $[0, T]$. The recipe is as follows.

We start with the measure \mathbb{Q} for which W_t^* is a Brownian motion. Given W_t^* we then construct the process H_t by use of formula (132), and the process Λ_t^* by use of formula (133), and the process \hat{M}_t by use of formula (134). Thus we see that the wave function $|\psi_t\rangle$ along with all the related processes $\hat{R}_t, \hat{M}_t, \Lambda_t,$ and H_t can be explicitly constructed as functions of W_t^* and t . The physical measure \mathbb{P} constructed by use of Λ_t^* is then used for the calculation of ensemble averages. In particular, letting \mathbb{E} denote the expectation with respect to the physical measure \mathbb{P} , it follows from (134) that

$$\mathbb{E}[X_t] = \mathbb{E}^{\mathbb{Q}}[\Lambda_t^* X_t] \tag{147}$$

for any \mathcal{F}_t -measurable random variable X_t .

For example, suppose \hat{G} is an observable that does not necessarily commute with the Hamiltonian \hat{H} , and we wish to calculate the ensemble average of the expectation value $\langle \psi_t | \hat{G} | \psi_t \rangle$. Then by use of (147) we have

$$\begin{aligned} \mathbb{E}[\langle \psi_t | \hat{G} | \psi_t \rangle] &= \mathbb{E}^{\mathbb{Q}}[\Lambda_t^* \langle \psi_t | \hat{G} | \psi_t \rangle] \\ &= \mathbb{E}^{\mathbb{Q}}[\Lambda_t^* \langle \psi_0 | \hat{U}_t^\dagger \hat{R}_t \hat{G} \hat{R}_t \hat{U}_t | \psi_0 \rangle] \\ &= \mathbb{E}^{\mathbb{Q}}[\langle \psi_0 | e^{i\hat{H}t + \frac{1}{2}\sigma \hat{H} W_t^* - \frac{1}{4}\sigma^2 \hat{H}^2 t} \hat{G} e^{-i\hat{H}t + \frac{1}{2}\sigma \hat{H} W_t^* - \frac{1}{4}\sigma^2 \hat{H}^2 t} | \psi_0 \rangle] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\sum_{m,n} G_{mn} e^{i(E_m - E_n)t + \frac{1}{2}\sigma(E_m + E_n)W_t^* - \frac{1}{4}(E_m^2 + E_n^2)\sigma^2 t}\right] \end{aligned} \tag{148}$$

where the matrix elements G_{mn} are given by

$$G_{mn} = \langle \psi_0 | \hat{P}_m \hat{G} \hat{P}_n | \psi_0 \rangle. \tag{149}$$

Since W_t^* is normally distributed with mean zero and variance t with respect to the \mathbb{Q} -measure, the expectation in (148) can be readily computed. By use of the simple relation

$$\mathbb{E}^{\mathbb{Q}}[e^{\alpha W_t^*}] = e^{\frac{1}{2}\alpha^2 t} \tag{150}$$

which holds for any constant α , we see that

$$\mathbb{E}^{\mathbb{Q}}[e^{\frac{1}{2}\sigma(E_m + E_n)W_t^* - \frac{1}{4}(E_m^2 + E_n^2)\sigma^2 t}] = e^{-\frac{1}{8}\sigma^2(E_m - E_n)^2 t}. \tag{151}$$

As a consequence we deduce that

$$\mathbb{E}[\langle \psi_t | \hat{G} | \psi_t \rangle] = \sum_{m,n} \bar{G}_{mn} e^{i(E_m - E_n)t - \frac{1}{8}\sigma^2(E_m - E_n)^2 t} \tag{152}$$

where

$$\begin{aligned} \bar{G}_{mn} &= \mathbb{E}[\langle \psi_0 | \hat{P}_m \hat{G} \hat{P}_n | \psi_0 \rangle] \\ &= \text{Tr} \hat{\rho}_0 (\hat{P}_m \hat{G} \hat{P}_n) \\ &= \text{Tr} \hat{G} (\hat{P}_n \hat{\rho}_0 \hat{P}_m) \end{aligned} \tag{153}$$

and $\hat{\rho}_0$ is the density matrix corresponding to the random initial state. This result is consistent with our earlier expression (109) for the solution of the Lindblad equation, and illustrates the fact that the change of measure technique is indeed highly effective as a calculational tool for quantum state reduction models. For related work, see, e.g. [32].

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References

- [1] von Neumann J 1932 *Mathematische Grundlagen der Quantenmechanik* (Berlin: Springer) (Engl. Transl. R T Beyer 1971 *Mathematical Foundations of Quantum Mechanics*) (Princeton: Princeton University Press)
- [2] Lüders G 1951 *Ann. Phys., Lpz.* **8** 322
- [3] Gisin N 1984 *Phys. Rev. Lett.* **52** 1657
Gisin N 1989 *Helv. Phys. Acta* **62** 363
- [4] Diosi L 1988 *J. Phys. A: Math. Gen.* **21** 2885
Diosi L 1988 *Phys. Lett. A* **129** 419
Diosi L 1988 *Phys. Lett. A* **132** 233
- [5] Ghirardi G C, Pearle P and Rimini A 1990 *Phys. Rev. A* **42** 78
- [6] Percival I 1994 *Proc. R. Soc. A* **447** 189
- [7] Hughston L P 1996 *Proc. R. Soc. A* **452** 953
- [8] Adler S L and Horwitz L P 2000 *J. Math. Phys.* **41** 2485
- [9] Brody D C and Hughston L P 2001 *Proc. R. Soc. A* to be published
(Brody D C and Hughston L P 2000 *Preprint quant-ph/0011125*)
- [10] Pearle P 2000 *Open Systems and Measurement in Relativistic Quantum Theory* ed H-P Breuer and F Petruccione (Berlin: Springer)
- [11] Ghirardi G C 2000 *Quantum Reflections* ed J Ellis and D Amati (Cambridge: Cambridge University Press)
- [12] Hughston L P, Jozsa R and Wootters W K 1993 *Phys. Lett. A* **183** 14
- [13] Adler S L and Brun T A 2001 *J. Phys. A: Math. Gen.* **34** 4797
- [14] Percival I C 1994 *J. Phys. A: Math. Gen.* **27** 1003
- [15] Schack R, Brun T A and Percival I C 1995 *J. Phys. A: Math. Gen.* **28** 5401
- [16] Ikeda N and Watanabe S 1981 *Stochastic Differential Equations and Diffusion Processes* (Amsterdam: North-Holland)
- [17] Revuz D and Yor M 2001 *Continuous Martingales and Brownian Motion* 3rd edn Corrected 2nd print (Berlin: Springer)
- [18] Meyer P A 1966 *Probability and Potentials* (Waltham, MA: Blaisdell Publishing Company)
- [19] Isham C J 1995 *Lectures on Quantum Theory* (London: Imperial College Press)
- [20] Pauli W 1933 Die allgemeinen Prinzipien der Wellenmechanik *Handbuch der Physik* vol 24, ed H Geiger and K Scheel (Berlin: Springer) pp 83–272
- [21] Wheeler J A and Zurek W H (ed) 1983 *Quantum Theory and Measurement* (Princeton: Princeton University Press)
- [22] Gorini V, Kossakowski A and Sudarshan E C G 1976 *J. Math. Phys.* **17** 821
- [23] Lindblad G 1976 *Commun. Math. Phys.* **48** 119
- [24] Parthasarathy K R 1992 *An Introduction to Quantum Stochastic Calculus* (Basel: Birkhäuser)
- [25] Carmichael H J 1993 *An Open Systems Approach to Quantum Optics* (Berlin: Springer)
- [26] Diosi L, Gisin N, Halliwell J and Percival I C 1995 *Phys. Rev. Lett.* **74** 203
- [27] Brun T A 1997 *Phys. Rev. Lett.* **78** 1833
- [28] Adler S L 2000 *Phys. Lett. A* **265** 58
- [29] Gardiner C W and Zoller P 2000 *Quantum Noise* 2nd edn (Berlin: Springer)
- [30] Wiseman H M and Diosi L 2001 *J. Chem. Phys.* **268** 91
- [31] Karatzas I and Shreve S E 1991 *Brownian Motion and Stochastic Calculus* 2nd edn (Berlin: Springer)
- [32] Gatarek D and Gisin N 1991 *J. Math. Phys.* **32** 2152
Diosi L, Gisin N and Strunz W T 1998 *Phys. Rev. A* **58** 1699